Universal Journal of Applied Mathematics & Computation 7 (2019), 20-34
www.papersciences.com

Qualitative Analysis of a Predator-Prey Model in the Presence of Additional Food to Predator & Constant-Yield Predator Harvesting

Manoj Kumar Singh and Poonam
Department of Mathematics & Statistics,
Banasthali Vidyapith, Rajasthan-304022, India
s.manojbbau@gmail.com, raopoonamamit@gmail.com

Abstract
In the present article, a predator-prey model with additional food and constant yield harvesting rate to predator is considered. It is assumed that additional food is not dynamic in nature, but available at a specific constant level either by the nature or by an external agency. The local stability of the equilibrium points of the model has been investigated. Further, it is shown that the model undergoes to different kind of bifurcations including Hopf bifurcation, Transcritical bifurcation, Saddle-Node bifurcation and Bogdanov-Takens bifurcation. The numerical simulation has been done which is in good agreement to the analytical findings.

2010 Mathematics Subject Classification: 92B05, 35B35, 34C23, 37G10.
Key Word and Phrases
Predator-Prey Model, Stability, Bifurcation, Additional Food, Harvesting.

1. Introduction

The predator-prey interactions in ecosystem are very complex, and in order to develop a mathematical model, which explains the real life situation best, many changes and development have been made by researcher after the pioneering work Lotka-Volterra predator-prey model proposed by A.Lotka [1] in 1925 and V.Volterra [2] in 1926 independently. The harvesting of the marine and wild species is one of the most useful as well as dangerous interference by humans in the ecosystem because one side it provides food to a large population but the overexploitation may damage the ecosystem. Thus the management of renewable Biological requires a scientific analysis. The analysis of harvesting in predator-prey system started with the work proposed by C.W.Clark [3], the problem of combined harvesting of two fish species which are ecologically independent and growing logistically was studied. The global behavior of a predator-prey system with constant rate predator harvesting and constant rate prey harvesting was studied by F.Brauer and A.C.Soudack [4]-[5]. J.R.Beddington and J.G.Cooke [6] studied a Leslie-Gower type predator-prey system in which preys are harvested at constant-yield rate and predators are harvested with constant-effort rate. In the same paper they also studied same system with constant yield harvesting on both the prey and predators. D.Xiao and S.Ruan [7] discussed the Bogdanov-Takens bifurcation for a predator-prey model with Holling-Type II functional response and constant rate predator harvesting. D.Xiao and L.S.Jennings [8] studied a predator-prey model with ratio-dependent type functional response in the presence of constant harvesting in prey species, while M.Xiao et al. [9] studied the same model but for constant predator harvesting and found the different dynamics. C.R.Zhu and K.Q.Lan [10] studied a Leslie-Gower predator-prey model with constant harvesting in prey only and studied phase portraits near the interior equilibrium. They also proved that the nature of predator free equilibrium depend upon the choices of the parameters while the interior positive equilibrium in the first quadrant are saddles, stable or unstable nodes, foci, centres, saddle-nodes or cusps. Y.Gong and J.Huang [11] studied the Bogdanov-Takens bifurcation for this model. J.Huang et al. [12] studied a predator-prey model with constant yield predator harvesting and showed that for some parametric conditions the system has cusps of codimension 2 and 3. The conditions for which the system has repelling B-T bifurcation and attracting B-T bifurcation are obtained.

A number of species in the ecosystem exist which are migratory whose special scale is much longer than the habitat occupied by some of their prey, and so, for such types of species and alternative prey is required R.D.Holt and J.H.Lawton [13]. The additional food to predator is modelled mathematically as three species; one predator-two (non-interacting) prey system is available in M. van Baalen [14], J.T.Wootton [15] and J.D. Harwood and J.J.Obrycki [16]. An important result of these models is that the non-reproducing additional food (referred to as additional prey) to predator enhances the predator density which decreases the density of the target prey required R.D.Holt and J.H.Lawton [13], M. van Baalen [14]. But there are some practical work also available in literature, indicate that provision of additional food to predators need not always increase target predation R.D.Holt and J.H.Lawton [13], M. van Baalen [14], J.T.Wootton [15], J.D. Harwood and J.J.Obrycki [16]. P.D.Spencer and J.S.Collie in [17] studied a two–species population model in which the predator is partially coupled to the prey in the presence of harvesting and intraspecific competition into predatory fish. P.D.N.Srinivasu et al. [18] have proposed another two dimensional predator-prey model with additional food to the predator and discussed the effect of both high and low quality of food. M.Sen et al. [19] studied the global dynamics for this model in the presence of constant yield harvesting in predator.

T.K.Kar and B.Ghosh [20] studied a two species predator-prey model in which the predator is partially coupled with alternative prey and harvesting efforts applied to both the species. In this model, it is assumed that the additional food is not dynamic in nature, but available at a specific constant level either by the nature or by an external agency, they provided three examples for which the proposed model fits best. The purpose of this paper is to study the stability and bifurcation analysis for the model proposed by T.K.Kar and B.Ghosh [20] in the presence of constant yield harvesting in predator species. This work presents management strategies that manipulate the supply amount of additional food and rate of harvesting of the predator for the benefit of biological control of the system.

2. Model Equations

We consider the following bidimensional predator-prey system with constant-yield harvesting in predator species

$$\begin{cases} \frac{dN}{dT} = rN\left(1-\frac{N}{K}\right) - \frac{\alpha NP}{\theta+N}, \\ \frac{dP}{dT} = \frac{\alpha\beta NP}{\theta+N} - \mu P - H, \end{cases} \tag{2.1}$$

with the initial conditions $N(0) > 0, P(0) > 0$, where $N(T)$ and $P(T)$ are prey and predator density at time T and r, K, α, θ, μ, β and H are positive parameters which represent intrinsic growth rate of prey, carrying capacity of prey in the absence of predator, capturing rate of the predator, conversion efficiency of predators, the extent to which the environment provides protection for prey, natural mortality rate and harvesting coefficient, respectively. F.Brauer and A.C.Soudack [4] studied the global behavior of system (2.1) for some parameter values by numerical simulations while D.Xiao and S.Ruan [7] studied the bifurcation analysis of the system (2.1).

In this article, it has been assumed that the predator is provided with additional food of biomass A (a time independent positive constant) which is distributed uniformly in habitat. Then the following system describes the predator-prey dynamics in the presence of additional food to predator and constant-yield predator harvesting:

$$\begin{cases} \frac{dN}{dT} = rN\left(1-\frac{N}{K}\right) - \frac{\alpha ANP}{\theta+N}, \\ \frac{dP}{dT} = \frac{\alpha\beta ANP}{\theta+N} + (1-A)P - \mu P - H, \end{cases} \tag{2.2}$$

If $A = 0$ the prey and predator will grow independently, that is, the system (2.2) becomes decoupled system. For $A = 1$ the predator depends only on the available food (focal prey N) and the system (2.2) will be similar to the system (2.1). Moreover, system (2.2) is dynamically equivalent to the system (2.1) whenever $A > 1$ or $1 - A < \mu$. Thus, our interest is to study the dynamical behavior of the system (2.2) for $0 < A < 1$ and $A + \mu < 1$. On introducing the non-dimensional variables: $N = Kx,\ P = y,\ T = \frac{t}{r}$, the system (2.2) reduces to:

$$\begin{cases} \frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{x}(1-\mathrm{x}) - \frac{\mathrm{axy}}{\mathrm{m+x}} = \mathrm{xf_1(x,y)}, \\ \frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{bxy}}{\mathrm{m+x}} + \delta\mathrm{y} - \mathrm{h} = \mathrm{f_2(x,y)}, \end{cases} \tag{2.3}$$

with the initial conditions: $x(0) > 0, y(0) > 0$,

where $a = \frac{\alpha A}{rK}$, $m = \frac{\theta}{K}$, $b = \frac{\beta\alpha A}{r}$, $h = \frac{H}{r}$, $\delta = \frac{1-A-\mu}{r} > 0$, $f_1(x,y) = 1 - x - \frac{ay}{m+x}$ and $f_2(x,y) = \frac{bxy}{m+x} + \delta y - h$.

3. Equilibrium Points and Qualitative Analysis

The equilibrium points of the system (2.3) are the non-negative real solutions of the prey zero growth isoclines:

$$\mathrm{xf_1(x,y)} = 0, \tag{3.1}$$

and predator zero growth isoclines:

$$\mathrm{f_2(x,y)} = 0. \tag{3.2}$$

The following two type of equilibrium points for the system (3) exist:

(a) *Axial equilibrium points:* The axial equilibrium points of the system (2.3) are only the points of intersection of the curves $x = 0$ and $g(x,y) = 0$ which is $E_1 = \left(0, \frac{h}{\delta}\right)$.

(b) *Interior equilibrium points:* The interior equilibrium points of the system (2.3) are the intersection points, $E_1^* = (x_1^*, y_1^*)$ and $E_2^* = (x_2^*, y_2^*)$ of the curves $f(x,y) = 0$ and $g(x,y) = 0$, and the abscissa of the equilibrium points are the solutions of the quadratic equation

$$(\mathrm{b}+\delta)\mathrm{x}^2 - (\mathrm{b}+\delta-\delta\mathrm{m})\mathrm{x} + \mathrm{ah} - \delta\mathrm{m} = 0, \tag{3.3}$$

while the ordinance are given by $y_k^* = \frac{h(m+x_k^*)}{\delta\mathrm{m}+(\mathrm{b}+\delta)x_k^*}$, $k = 1, 2$, where:

$$x_k^* = \frac{b+\delta-\delta\mathrm{m}+(-1)^{\mathrm{k}+1}\sqrt{(b+\delta+\delta\mathrm{m})^2-4ah(b+\delta)}}{2(\mathrm{b}+\delta)},\ \ k = 1, 2.$$

If $m < 1 + \frac{b}{\delta}$, the quadratic equation (3.3), has two positive real roots x_1^* and x_2^* whenever $\frac{\delta\mathrm{m}}{a} < h < \frac{(b+\delta+\delta\mathrm{m})^2}{4a(b+\delta)}$; a double positive real root $\bar{x} = \frac{b+\delta-\delta\mathrm{m}}{2(b+\delta)}$, whenever $\frac{\delta\mathrm{m}}{a} < h = \frac{(b+\delta+\delta\mathrm{m})^2}{4a(b+\delta)}$; one positive real root $x^* = \frac{b+\delta-\delta\mathrm{m}}{b+\delta}$, whenever $h = \frac{\delta\mathrm{m}}{a}$ and one positive real root $x_3 = \frac{b+\delta-\delta\mathrm{m}+\sqrt{(b+\delta+\delta\mathrm{m})^2-4ah(b+\delta)}}{2(\mathrm{b}+\delta)}$, whenever $h < \frac{\delta\mathrm{m}}{a}$. If $m > 1 + \frac{b}{\delta}$,the quadratic equation (3.3), has one positive real roots $x_4 = \frac{b+\delta-\delta\mathrm{m}+\sqrt{(b+\delta+\delta\mathrm{m})^2-4ah(b+\delta)}}{2(\mathrm{b}+\delta)}$, whenever $h < \frac{\delta\mathrm{m}}{a}$, has no equilibrium point whenever $\frac{\delta\mathrm{m}}{a} < h$.

On summarizing the above discussion, the number and location of equilibrium points of system (2.3) can be described by the following:

Lemma 3.1

(i) The system (2.3) always has an axial equilibrium point $\mathrm{E_1} = \left(0, \frac{\mathrm{h}}{\delta}\right)$.

(ii) If $m < 1 + \frac{b}{\delta}$, the system (2.3) has:

a) no interior equilibrium point whenever $h > \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}$.

b) two positive interior equilibrium point $E_1^* = (x_1^*, y_1^*)$, $E_2^* = (x_2^*, y_2^*)$ if $\frac{\delta m}{a} < h < \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}$.

c) one positive interior equilibrium point $\overline{E} = (\overline{x}, \overline{y})$ if $\frac{\delta m}{a} < h < \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}$.

d) one positive interior equilibrium point $E^* = (x^*, y^*)$ if $h = \frac{\delta m}{a}$.

e) one positive interior equilibrium point $E_3 = (x_3, y_3)$ if $h < \frac{\delta m}{a}$.

(iii) If $m > 1 + \frac{b}{\delta}$, the system (2.3) has,

a) no interior equilibrium point whenever $\frac{\delta m}{a} < h$.

b) one positive interior equilibrium point $E_3 = (x_4, y_4)$ if $h < \frac{\delta m}{a}$.

Now, we discuss the dynamics of system (2.3) in the neighbourhood of each equilibrium point by using linearization technique.

Theorem 3.1

a) The axial equilibrium point E_1 of the system (2.3) is an unstable hyperbolic node if $ah < \delta m$ and hyperbolic saddle if $ah > \delta m$.

b) The interior equilibrium point E_1^* of the system (2.3), if exists, is an unstable hyperbolic saddle.

c) The interior equilibrium point E_2^* of the system (2.3), if exists, is asymptotically stable if $\delta + \frac{x_2^*}{m+x_2^*}(1 + b - m - 2x_2^*) < 0$, is saddle if $\delta + \frac{x_2^*}{m+x_2^*}(1 + b - m - 2x_2^*) > 0$, is a weak focus or a center if $\delta + \frac{x_2^*}{m+x_2^*}(1 + b - m - 2x_2^*) = 0$.

d) The interior equilibrium points E^*, E_3 and E_4 of the system (2.3), if exist, are always a saddle point.

e) The interior equilibrium point $\overline{E}$ of the system (2.3), if exists, is a degenerate singularity.

Proof: a) The Jacobian matrix of the system (2.3) at the axial equilibrium point E_1 is:

$$J_{E_1} = \begin{bmatrix} 1 - \frac{ah}{\delta m} & 0 \\ \frac{bh}{\delta m} & \delta \end{bmatrix},$$

which confirms that the axial equilibrium point E_1 is an unstable hyperbolic node if $ah < \delta m$ and a hyperbolic saddle if $ah > \delta m$.

b) The Jacobian matrix of the system (2.3) at the equilibrium point E_1^* is:

$$J_{E_1^*} = \begin{bmatrix} x_1^*\left(-1 + \frac{ay_1^*}{(m+x_1^*)^2}\right) & -\frac{ax_1^*}{m+x_1^*} \\ \frac{bmy_1^*}{(m+x_1^*)^2} & \frac{bx_1^*}{m+x_1^*} + \delta \end{bmatrix}.$$

The determinant of the Jacobian matrix $J_{E_1^*}$, $\det(J_{E_1^*}) = -\frac{x_1^*}{m+x_1^*}\sqrt{(b + \delta + \delta m)^2 - 4ah(b + \delta)} < 0$,

confirms that the point E_1^* is a hyperbolic saddle.

c) The Jacobian matrix of the system (2.3) at the equilibrium point E_2^* is:

$$J_{E_2^*} = \begin{bmatrix} x_2^*\left(-1+\frac{ay_2^*}{(m+x_2^*)^2}\right) & -\frac{ax_2^*}{m+x_2^*} \\ \frac{bmy_2^*}{(m+x_2^*)^2} & \frac{bx_1^*}{m+x_2^*}+\delta \end{bmatrix}.$$

$\det(J_{E_2^*}) = \frac{x_2^*}{m+x_2^*}\sqrt{(b+\delta+\delta m)^2 - 4ah(b+\delta)} > 0$, and the trace of $J_{E_2^*}$ is $\mathrm{tr}(J_{E_2^*}) = \delta + \frac{x_2^*}{m+x_2^*}(1+b-m-2x_2^*) < 0$. Therefore, if $\delta + \frac{x_2^*}{m+x_2^*}(1+b-m-2x_2^*) < 0$, the equilibrium point E_2^* is asymptotically stable, if $\delta + \frac{x_2^*}{m+x_2^*}(1+b-m-2x_2^*) > 0$, the equilibrium point E_2^* is unstable and if $\delta + \frac{x_2^*}{m+x_2^*}(1+b-m-2x_2^*) = 0$, which implies that the equilibrium point E_2^* is either a weak focus or a center as the eigenvalues of the Jacobian matrix $J_{E_2^*}$ are purely imaginary.

d) The Jacobian matrix of the system (2.3) at the equilibrium point E^* is:

$$J_{E^*} = \begin{bmatrix} x^*\left(-1+\frac{ay^*}{(m+x^*)^2}\right) & -\frac{ax^*}{(m+x^*)} \\ \frac{bmy^*}{(m+x^*)^2} & \frac{bx^*}{m+x^*}+\delta \end{bmatrix}.$$

$\det(J_{E^*}) = -\frac{(b+\delta-\delta m)^2}{b+\delta+\delta m} < 0$. Thus, the equilibrium point $E^*$is a saddle point.

The Jacobian matrix of the system (2.3) at the equilibrium point E_3 is:

$$J_{E_3} = \begin{bmatrix} x_3\left(-1+\frac{ay_3}{(m+x_3)^2}\right) & -\frac{ax_3}{(m+x_3)} \\ \frac{bmy_3}{(m+x_3)^2} & \frac{bx_3}{m+x_3}+\delta \end{bmatrix}.$$

$\det(J_{E_3}) = -\frac{x_3}{m+x_3}\sqrt{(b+\delta+\delta m)^2 - 4ah(b+\delta)} < 0$. Thus, the point E_3 is a saddle point.

The Jacobian matrix of the system (2.3) at the equilibrium point E_4 is:

$$J_{E_4} = \begin{bmatrix} x_4\left(-1+\frac{ay_4}{(m+x_4)^2}\right) & -\frac{ax_4}{(m+x_4)} \\ \frac{bmy_4}{(m+x_4)^2} & \frac{bx_4}{m+x_4}+\delta \end{bmatrix}.$$

$\det(J_{E_4}) = -\frac{x_3}{m+x_4}\sqrt{(b+\delta+\delta m)^2 - 4ah(b+\delta)} < 0$. Thus, the point E_4 is a saddle point.

e) The Jacobian matrix of the system (2.3) at the equilibrium point $\bar{E}$ is

$$J_{\bar{E}} = \begin{bmatrix} \bar{x}\left(-1+\frac{a\bar{y}}{(m+\bar{x})^2}\right) & -\frac{a\bar{x}}{(m+\bar{x})} \\ \frac{bm\bar{y}}{(m+\bar{x})^2} & \frac{b\bar{x}}{m+\bar{x}}+\delta \end{bmatrix}.$$

$\det(J_{\bar{E}}) = 0$ implies that the point $\bar{E}$ is degenerate singularity and may has complicated properties.

It has been proved in the theorem 3.1 that the unique interior equilibrium point $\bar{E}$ is a degenerate singularity and hence may have complicated properties. Now, we discuss the behavior of interior equilibrium point $\overline{E}$.

Theorem 3.2

If the unique interior equilibrium point $\bar{E}$ exists, it is:

a) a saddle node whenever $\alpha_{10} + \beta_{01} \neq 0$.

b) a cusp of codimension 2 whenever $\alpha_{10}+\beta_{01}=0, \alpha_{10}\alpha_{20}+\alpha_{01}\beta_{20}-\alpha_{10}\beta_{11}-\frac{{\alpha_{10}}^2\alpha_{11}}{\alpha_{01}}\neq 0$ and $2\alpha_{20}-\frac{\alpha_{11}\alpha_{10}}{\alpha_{01}}+\beta_{11}\neq 0$.

Proof: Consider $\hat{x}=x-\bar{x}$, $\hat{y}=y-\bar{y}$. This transformation shifts the equilibrium point $\bar{E}$ of the system (2.3) to the origin and for the sake of convenience denote $\hat{x}$ as x and $\hat{y}$ as y, the system (2.3) can be rewritten as:

$$\begin{cases}\frac{dx}{dt}=\alpha_{10}x+\alpha_{01}y+\alpha_{20}x^2+\alpha_{11}xy+o^{3+}(x,y)\,,\\ \frac{dy}{dt}=\beta_{10}x+\beta_{01}y+\beta_{20}x^2+\beta_{11}xy+o^{3+}(x,y)\,,\end{cases} \tag{3.4}$$

where $\alpha_{10}=-\frac{bm\bar{x}}{(b+\delta)(m+\bar{x})}$, $\alpha_{01}=-\frac{a\bar{x}}{(m+\bar{x})}$, $\alpha_{20}=-1+\frac{m(1-\bar{x})}{(m+\bar{x})^2}$, $\alpha_{11}=-\frac{am}{(m+\bar{x})^2}$, $\beta_{10}=\frac{bm(1-\bar{x})}{a(m+\bar{x})}$, $\beta_{01}=\frac{b\bar{x}}{(m+\bar{x})}+\delta$, $\beta_{20}=-\frac{bm(1-\bar{x})}{a(m+\bar{x})^2}$, $\beta_{11}=\frac{bm}{(m+\bar{x})^2}$ and $o^{3+}(x,y)$ are the terms involving order three and greater.

If $\alpha_{10}+\beta_{01}\neq 0$, then $tr(J_{\bar{E}})\neq 0$ but $\det(J_{\bar{E}})=0$. Hence $\bar{E}$ is a saddle node. Further, $tr(J_{\bar{E}})=0$, whenever $\alpha_{10}+\beta_{01}=0$. Now, on using the transformation $y_1=x$, $y_2=\alpha_{10}x+\alpha_{01}y$ with the parametric condition $\det(J_{\bar{E}})=0$, the system (3.4) reduces to the following system

$$\begin{cases}\frac{dy_1}{dt}=y_2+\overline{\alpha_{20}}y_1^2+\overline{\alpha_{11}}y_1y_2+o^{3+}(y_1,y_2),\\ \frac{dy_2}{dt}=\overline{\beta_{20}}y_1^2+\overline{\beta_{11}}y_1y_2+o^{3+}(y_1,y_2),\end{cases} \tag{3.5}$$

where:

$\overline{\alpha_{20}}=\alpha_{20}-\frac{\alpha_{11}\alpha_{10}}{\alpha_{01}}$, $\overline{\alpha_{11}}=\frac{\alpha_{11}}{\alpha_{01}}$, $\overline{\beta_{20}}=\alpha_{10}\alpha_{20}+\alpha_{01}\beta_{20}-\alpha_{10}\beta_{11}-\frac{{\alpha_{10}}^2\alpha_{11}}{\alpha_{01}}$, $\overline{\beta_{11}}=\beta_{11}+\frac{\alpha_{10}\alpha_{11}}{\alpha_{01}}$.

On using the transformation $z_1=y_1-\frac{1}{2}\overline{\alpha_{11}}y_1^2$, $z_2=y_2+\overline{\alpha_{20}}y_1^2$, the system (3.5) reduces to

$$\begin{cases}\frac{dz_1}{dt}=z_1+o^{3+}(z_1,z_2),\\ \frac{dz_2}{dt}=\overline{\beta_{20}}z_1^2+(2\overline{\alpha_{20}}+\overline{\beta_{11}})z_2+o^{3+}(z_1,z_2).\end{cases} \tag{3.6}$$

Finally, using the transformation $X=z_1$, $Y=z_2+o^{3+}(z_1,z_2)$, the system (3.6) reduces to

$$\begin{cases}\frac{dX}{dt}=Y\\ \frac{dY}{dt}=\overline{\beta_{20}}X^2+\left(\overline{2\alpha_{20}}+\overline{\beta_{11}}\right)XY+o^{3+}(X,Y).\end{cases} \tag{3.7}$$

If $\overline{\beta_{20}}\neq 0$ and $\overline{2\alpha_{20}}+\overline{\beta_{11}}\neq 0$, the system (3.7) is a cusp of codimension 2 at the origin in XY plane, i.e. $\bar{E}$ in xy-plane is a cusp of codimension 2. These conditions are known as non-degeneracy condition of a cusp of codimension 2.

4. Bifurcation Analysis

In this section, we discuss the bifurcations, which occur in the system (2.3). In previous section, it is shown that for certain parametric conditions, some of the equilibrium points may be hyperbolic or degenerate singularities, and hence, there is a possibility of bifurcation.

4.1 Hopf Bifurcation

In Theorem 3.1, it is proved that the interior equilibrium point E_1^* is always a saddle point while E_2^*, is a weak focus or a center if $\delta + \frac{x_2^*}{m+x_2^*}(1+b-m-2x_2^*) = 0$. A Hopf bifurcation occurs where a periodic orbit is created as the stability of the equilibrium point E_2^*, loses.

Theorem 4.1

The system (2.3) undergoes a hopf bifurcation with respect to bifurcation parameter δ around the point E_2^*, if $\delta + \frac{x_2^*}{m+x_2^*}(1+b-m-2x_2^*) = 0$ and:

1) an unstable limit cycle arises around the point E_2^*, if $(a,b,m,h,\delta) \in H_1$,
2) a stable limit cycle arises E_2^*, if $(a,b,m,h,\delta) \in H_2$,

Proof: If δ be the bifurcation parameter, then threshold magnitude $\delta = \delta^{|hf|}$ exist, which satisfies $\det(J_{E_2^*}) > 0$ and $\mathrm{tr}(J_{E_2^*}) = 0$ and hence both the eigenvalue of the system (2.3) will be purely imaginary. Also:

$$\left[\frac{d}{d\delta}\left(\mathrm{Tr}(J_{E_2^*})\right)\right]_{\delta=\delta^{|hf|}} = 1 + \frac{1}{m+x_2^*}\left(2x_2^* + \frac{\delta m}{x_2^*}\right)\frac{\left(2ah(b+\delta) + bm(b+\delta+\delta m) + bm\sqrt{(b+\delta+\delta m)^2 - 4ah(b+\delta)}\right)}{2(b+\delta)^2\sqrt{(b+\delta+\delta m)^2 - 4ah(b+\delta)}} \neq 0$$

which is the transversality condition of the hopf bifurcation. This guarantees the existence of hopf bifurcation (see, L.Perko [21]).

Now, to study the stability of limit cycle we compute the first Lyapunov number σ at interior equilibrium point $E_2^*(x_2^*, y_2^*)$ of the system (2.3) using the procedure as given in L.Perko [21].
Let $x = u - x_2^*$, $y = v - y_2^*$. Then the system (2.3), in the vicinity of the origin, can be written as:

$$\frac{du}{dt} = a_{10}u + a_{01}v + a_{20}v^2 + a_{30}u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + P(u,v),$$

$$\frac{dv}{dt} = b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + Q(u,v).$$

where:

$a_{10} = x_2^*\left(-1 + \frac{ay_2^*}{(m+x_2^*)^2}\right)$, $a_{01} = -\frac{ax_2^*}{m+x_2^*}$, $a_{20} = \left(-1 + \frac{amy_2^*}{(m+x_2^*)^3}\right)$, $a_{11} = -\frac{am}{(m+x_2^*)^2}$, $a_{02} = 0$, $a_{30} = -\frac{amy}{(m+x_2^*)^4}$, $a_{21} = \frac{am}{(m+x_2^*)^3}$, $a_{12} = 0$, $a_{03} = 0$, $b_{10} = \frac{bmy_2^*}{(m+x_2^*)^2}$, $b_{01} = \frac{bx_2^*}{m+x_2^*} + \delta^{|hf|}$, $b_{20} = -\frac{bmy_2^*}{(m+x_2^*)^3}$, $b_{11} = \frac{bm}{(m+x_2^*)^2}$, $b_{02} = 0$, $b_{30} = \frac{bmy_2^*}{(m+x_2^*)^4}$ $b_{21} = -\frac{bm}{(m+x_2^*)^3}$, $b_{12} = 0$, $b_{03} = 0$,
$P(u,v) = \sum_{i+j=4}^{\infty} a_{ij}\, u^i v^j$ and $Q(u,v) = \sum_{i+j=4}^{\infty} b_{ij}\, u^i v^j$.

Hence the first Lyapunov number σ for the planer system is:

$$\sigma = -\frac{3\pi}{2\Delta^{3/2}a_{01}}\{[a_{10}b_{10}(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02})$$

$$+ b_{10}^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}(b_{02}^2 - a_{20}a_{02}) - a_{01}^2(2a_{20}b_{20} + b_{11}b_{20})$$

$$+ (a_{10}b_{10} - 2a_{10}^2)(b_{11}b_{02} - a_{11}a_{20})] - (a_{10}^2 + a_{01}b_{10})[3(b_{10}b_{03} - a_{01}a_{30}) + 2a_{10}(a_{21} + b_{12})$$

$$+ (b_{10}a_{12} - a_{01}b_{21})]\},$$

where:

$$\Delta = \frac{x_2^*}{m+x_2^*}\sqrt{(b+\delta+\delta m)^2 - 4ah(b+\delta)}.$$

Thus, the subcritical hopf bifurcation surface of the system (2.3) is:

$$H_1 = \{(a, b, m, h, \delta): \sigma > 0, m < 1 + \frac{b}{\delta}, \frac{\delta m}{a} < h < \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}\}.$$

Similarly, supercritical hopf bifurcation surface of the system (2.3) is:

$$H_2 = \left\{(a, b, m, h, \delta) : \sigma < 0, m < 1 + \frac{b}{\delta}, \frac{\delta m}{a} < h < \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}\right\}.$$

4.2 Transcritical Bifurcation

In Lemma 3.1, it is proved that the system (2.3) has always only one axial equilibrium point E_1. In Theorem 3.1, it is proved that the axial equilibrium point E_1 is an unstable hyperbolic node, if $ah < \delta m$, and a saddle, if $ah > \delta m$. If $ah = \delta m$, one eigenvalue of the Jacobian matrix J_{E_1} is zero and other is positive and also the interior equilibrium point E_1^* coincides with the axial equilibrium point E_1, whenever $b + \delta - \delta m > 0$. Thus, there is a chance of bifurcation around the axial equilibrium point E_1.

Here, Sotomayor's theorem has been applied to ensure that the system undergoes transcritical bifurcation at the equilibrium point E_1.

Theorem 4.2
The system (2.3) undergoes a transcritical bifurcation, if both conditions $ah = \delta m$ and $b + \delta - \delta m \neq 0$ hold.

Proof: Since $\det(J_{E_1}) = 0$, whenever if $ah = \delta m$. Thus, one eigenvalue of the Jacobian matrix J_{E_1} is zero. Suppose v and w are the eigenvectors of the Jacobian matrix J_{E_1} and $J_{E_1}^T$ corresponding to zero eigenvalue respectively. And are given by,

$$V = \begin{bmatrix} 1 \\ -\frac{bh}{\delta^2 m} \end{bmatrix}; \qquad W = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then, we have:

$$W^T F_\delta(E_1, \delta^{[tc]}) = 0,$$
$$W^T |DF_\delta(E_1, \delta^{[tc]}) v| = \frac{m}{ah},$$
$$W^T [D^2F(E_1, \delta^{[tc]})(v, v)] = \frac{2}{ah}(b + \delta - \delta m) \neq \text{ as either } (b + \delta - \delta m) > 0 \text{ or } (b + \delta - \delta m) < 0,$$

Where:

$$F_\delta(E_1, \delta^{[tc]}) = \begin{bmatrix} 0 \\ \frac{h}{\delta^{[tc]}} \end{bmatrix}; \; DF_\delta(E_1, \delta^{[tc]}) = \begin{bmatrix} \frac{1}{\delta^{[tc]}} & 0 \\ -\frac{b}{a\delta^{[tc]}} & 1 \end{bmatrix}; D^2F(E_1, \delta^{[tc]}) = \begin{bmatrix} -2 + \frac{2ay}{m^2} + \frac{2abh}{\delta^2 m^2} \\ \frac{-2by}{m^2} + \frac{2b^2h}{\delta^2 m^2} \end{bmatrix}.$$

Thus, the transversality conditions for transcritical bifurcation are satisfied.

4.3 Saddle-Node Bifurcation

From Lemma 3.1, we have if $m < 1 + \frac{b}{\delta}$, the system (2.3) has two positive interior equilibrium points E_1^* and E_2^*, if $\frac{\delta m}{a} < h < \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}$ and these two interior equilibrium points coincide with each other and a unique interior equilibrium point $\overline{E}$ is obtained, whenever $\frac{\delta m}{a} < h = \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}$. Also, the system (2.3) has no positive interior equilibrium points, if $h > \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}$. The annihilation of equilibrium is may be due to the occurrence of saddle-node bifurcation for interior equilibrium points, which takes place, when the harvesting parameter h crosses the critical value $h = h^{[SN]} = \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}$. Sotomayor's theorem has been used to ensure that the system (2.3) undergoes to saddle-node bifurcation, h is taken as bifurcation parameter**.**

Theorem 4.3

The system (2.3) undergoes a saddle-node bifurcation with respect to the bifurcation parameter h around the equilibrium point $\overline{E}$ if $m < 1 + \frac{b}{\delta} \frac{\delta m}{a} < h = \frac{(b+\delta+\delta m)^2}{4a(b+\delta)}$ and $\frac{\bar{x}}{m+\bar{x}}(1 + b - m - 2\bar{x}) + \delta < 0$.

Proof: Since $\det(J_{\overline{E}}) = 0$, therefore one eigenvalue of the Jacobian matrix $J_{\overline{E}}$ is zero. Further, If $\mathrm{tr}(J_{\overline{E}}) < 0$, then other eigenvalue has negative real part. Suppose v and w be the eigenvectors corresponding to zero eigenvalue matrix of $J_{\overline{E}}$ and $J_{\overline{E}}^T$ respectively, and are given by,

$$V = \begin{bmatrix} 1 \\ \frac{1-m-2\bar{x}}{a} \end{bmatrix}; \qquad W = \begin{bmatrix} 1 \\ \frac{-a(1-m-2\bar{x})}{bm(1-\bar{x})} \end{bmatrix}$$

Now we have,

$$W^T F_h(\overline{E}, h^{[SN]}) = -\frac{2a}{b+\delta+\delta m} \neq 0,$$

$$W^T[D^2F(\overline{E}, h^{[SN]})(V,V)] = -\frac{2\bar{x}}{m+\bar{x}} - \frac{4bm}{b+\delta+\delta m} < 0.$$

Where:

$$F_h(\overline{E}, h^{[SN]}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}; D^2F(\overline{E}, h^{[SN]}) = \begin{bmatrix} -2 + \frac{2am\bar{y}}{(m+\bar{x})^3} - \frac{2m(1-m-2\bar{x})}{(m+\bar{x})^2} \\ -\frac{2bm\bar{y}}{(m+\bar{x})^3} + \frac{2bm(1-m-2\bar{x})}{a(m+\bar{x})^2} \end{bmatrix}.$$

Thus, the transversality conditions for saddle-node bifurcation are satisfied. The above equation discussion can be summarized as,

The biological interpretation of the saddle node bifurcation is that when the rate of harvesting is more than the critical harvesting rate $h^{[SN]}$, then the prey species goes to extinction, which drives to extinction the predator species. However, the both prey and predator species always coexist in the form of a positive equilibrium for certain choices of initial values, whenever $0 < h \leq h^{[SN]}$.

4.4 Bogdanov-Taken Bifurcation

Theorem (4.3) confirms that the system (2.3) undergoes a saddle-node bifurcation at the equilibrium point $\overline{E}$ if exist, whenever $\det(J_{\overline{E}}) = 0$ and $\mathrm{tr}(J_{\overline{E}}) \neq 0$. Consider the case when $\mathrm{tr}(J_{\overline{E}})$ is also zero. These two parametric conditions imply that the Jacobian matrix $J_{\overline{E}}$ has a doubled zero eigenvalue. Thus, here is a chance of co-dimension 2 bifurcations (Bogdanov-Takens bifurcation). In theorem 3.2, it is shown that the equilibrium point $\overline{E}$ is a cusp of co-dimension 2 whenever:

$$a_{10}a_{20} + a_{01}b_{20} - a_{10}b_{11} - \frac{a_{10}^2 a_{11}}{a_{01}} \neq 0 \text{ and } 2a_{20} - \frac{a_{11}a_{10}}{a_{01}} + b_{11} \neq 0.$$

Now, we consider δ and h as the bifurcation parameter and reduce the system (2.3) into normal form of the Bogdanov-Takens bifurcation by employing a series of C^∞ change of coordinates in a small domain of (0, 0).

Theorem 4.4

The system (2.3) undergoes a Bogdanov-Takens bifurcation parameters δ and h around the equilibrium point $\overline{E}$, if exist whenever $\frac{b\bar{x}}{(m+\bar{x})} - \frac{bm\bar{x}}{(b+\delta)(m+\bar{x})} + \delta = 0$, $a_{10}a_{20} + a_{01}b_{20} - a_{10}b_{11} - \frac{a_{10}^2 a_{11}}{a_{01}} \neq 0$ and $2a_{20} - \frac{a_{11}a_{10}}{a_{01}} + b_{11} \neq 0$. Moreover three bifurcation curves in λ1λ2 plane exist and through the B-T point and they are given by,

Saddle-node curve: SN= $\{(\lambda_1,\lambda_2): \mu_1(\lambda_1,\lambda_2) = 0\}$,

Hopf bifurcation curve: H= $\{(\lambda_1,\lambda_2): \mu_2(\lambda_1,\lambda_2) = \frac{\Upsilon_{11}}{\sqrt{\Upsilon_{20}}}\sqrt{-\mu_1(\lambda_1,\lambda_2)}, \mu_2(\lambda_1,\lambda_2) < 0\}$,

Homoclinic bifurcation curve: HL =$\{(\lambda_1,\lambda_2): \mu_2(\lambda_1,\lambda_2) = \frac{5\Upsilon_{11}}{7\sqrt{\Upsilon_{20}}}\sqrt{-\mu_1(\lambda_1,\lambda_2)}, \mu_2(\lambda_1,\lambda_2) < 0\}$.

Proof: Suppose the bifurcation parameter δ and h vary in a small domain of Bogdanov-Takens point (in brief, BT-point) (h_0, β_0), and let $(h_0 + \lambda_1, \delta_0 + \lambda_2)$ be a point of neighborhood of the BT-point (h_0, δ_0) where λ_1, λ_2 are small. Thus, the system (2.3) reduces to:

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{axy}{m+x} \\ \frac{dy}{dt} = \frac{bxy}{m+x} + (\delta_0 + \lambda_1)y - h_0 - \lambda_2 \end{cases} \tag{4.1}$$

The system (4.1) is C^∞ smooth with respect to the variables x, y in a small neighborhood of (h_0, δ_0).

Define $x_1 = x - \bar{x}, x_2 = y - \bar{y}$, system (4.1) reduces to:

$$\begin{cases} \frac{dx_1}{dt} = a_{10}x_1 + a_{01}x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + +R_1(x_1,x_2), \\ \frac{dx_2}{dt} = b_{00} + b_{10}x_1 + b_{01}x_2 + b_{20}x_1^2 + b_{11}x_1x_2 + R_2(x_1,x_2), \end{cases} \tag{4.2}$$

where $a_{10} = -\frac{bm\bar{x}}{(b+\delta)(m+\bar{x})}$, $a_{01} = -\frac{a\bar{x}}{(m-\bar{x})}$, $a_{20} = -1 + \frac{m(1-\bar{x})}{(m+\bar{x})^2}$, $a_{11} = \frac{am}{(m+\bar{x})^2}$, $b_{00} = \lambda_1\bar{y} - \lambda_2$, $b_{10} = \frac{bm(1-\bar{x})}{a(m+\bar{x})}$, $b_{11} = \frac{bm}{(m+\bar{x})^2}$ and R_1, R_2 are the power series in (x_1, x_2) with powers $x_1^i x_2^j$ satisfying $i + j \geq 3$.

Let us introduce the affine transformation $y_1 = x_1, y_2 = ax_1 + bx_2$ in the system (4.2), we get:

$$\begin{cases} \frac{dy_1}{dt} = y_2 + \xi_{20}(\lambda)y_1^2 + \xi_{11}(\lambda)y_1y_2 + \overline{R_1}(y_1,y_2), \\ \frac{dy_2}{dt} = \eta_{00}(\lambda) + \eta_{10}(\lambda)y_1 + \eta_{01}(\lambda)y_2 + \eta_{20}(\lambda)y_1^2 + \eta_{11}(\lambda)y_1y_2 + \overline{R_2}(y_1,y_2), \end{cases} \tag{4.3}$$

where $\xi_{20}(\lambda) = a_{20} - \frac{a_{11}a_{10}}{a_{01}}$, $\xi_{11}(\lambda) = \frac{a_{11}}{a_{01}}$, $\eta_{00}(\lambda) = a_{01}b_{00}$, $\eta_{10}(\lambda) = -a_{01}\lambda_1$, $\eta_{01} = \lambda_1$,

$\eta_{20} = a_{10}a_{20} + a_{01}b_{20} - \frac{a_{10}}{a_{01}}(a_{10}a_{11} + a_{01}b_{11})$, $\eta_{11} = \frac{a_{11}a_{10} + a_{01}b_{11}}{a_{01}}$ and $\overline{R_1}, \overline{R_2}$ are the power series in (y_1, y_2) with powers $y_1^i y_2^j$ satisfying $i + j \geq 3$.

Consider the C^∞ change of coordinates in the small neighborhood of (0,0): $z_1 = y_1 - \frac{1}{2}\xi_{11}y_1^2$, $z_2 = y_2 + \xi_{20}y_1^2$, which transform the system (4.3) into:

$$\begin{cases} \frac{dz_1}{dt} = z_2 + \overline{\overline{R_1}}(z_1,z_2),, \\ \frac{dz_2}{dt} = \eta_{00} + \eta_{10}z_1 + \eta_{01}z_2 + \left(\frac{1}{2}\xi_{11}\eta_{10} - \eta_{01}\xi_{20} + \eta_{20}\right)z_1^2 + \left(2\xi_{20} + \eta_{11}\right)z_1z_2 + \overline{\overline{R_2}}(z_1,z_2). \end{cases}$$

(4.4)

Then, the system (4.4) reduces to:

$$\begin{cases} \frac{du_1}{dt} = u_2, \\ \frac{du_2}{dt} = \Upsilon_{00} + \Upsilon_{10}u_1 + \Upsilon_{01}u_2 + \Upsilon_{20}u_1^2 + \Upsilon_{11}u_1u_2 + F_1(u_1) + u_2F_2(u_1) + u_2^2F_3(u_1, u_2), \end{cases} \tag{4.5}$$

where $\Upsilon_{00} = \eta_{00}$, $\Upsilon_{10} = \eta_{10}$, $\Upsilon_{01} = \eta_{01}$, $\Upsilon_{20} = \frac{1}{2}\xi_{11}\eta_{10} - \eta_{01}\xi_{20} + \eta_{20}$, $\Upsilon_{11} = 2\xi_{20} + \eta_{11}$, F_1 and F_1 are the power series in w_1 with powers $w_1^{k_1}$ and $w_1^{k_2}$ and F_3 is a power series in and w_1 and w_2 involves terms like $w_1^i w_2^j$ satisfying $k_1 \geq 3, k_2 \geq 2$ and $i + j \geq 1$.

We have $\Upsilon_{20} \neq 0$ and $\Upsilon_{11} \neq 0$. Applying the Malgrange preparation theorem, one has:

$$\Upsilon_{00} + \Upsilon_{10}u_1 + \Upsilon_{20}u_1^2 + F_1(u_1) = \left(u_1^2 + \frac{\Upsilon_{10}}{\Upsilon_{20}}u_1 + \frac{\Upsilon_{00}}{\Upsilon_{20}}\right)B_1(u_1, \lambda),$$

with $B_1(0, \lambda) = \Upsilon_{20}$ and B_1 is a power series of w_1 whose coefficients depend on parameters (λ_1, λ_2).

Let:

$X_1 = u_1$, $X_2 = \frac{u_2}{\sqrt{\Upsilon_{20}}}$ and $d\tau = \sqrt{\Upsilon_{20}}dt$, then the system (4.5) reduces to:

$$\begin{cases} \frac{dX_1}{d\tau} = X_2, \\ \frac{dX_2}{d\tau} = \frac{\Upsilon_{00}}{\Upsilon_{20}} + \frac{\Upsilon_{10}}{\Upsilon_{20}}X_1 + \frac{\Upsilon_{01}}{\sqrt{\Upsilon_{20}}}X_2 + X_1^2 + \frac{\Upsilon_{11}}{\sqrt{\Upsilon_{20}}}X_1X_2 + P(X_1, X_2, \lambda), \end{cases} \tag{4.6}$$

where $P(X_1, X_2, 0)$ is a power series in (X_1, X_2) with powers $X_1^i X_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

Applying the parameter dependent affine transformation $Y_1 = X_1 + \frac{\Upsilon_{10}}{2\Upsilon_{20}}$, $Y_2 = X_2$ in the system (4.6) and using Taylor series expansion, we get:

$$\begin{cases} \frac{dY_1}{d\tau} = Y_2 \\ \frac{dY_2}{d\tau} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2)Y_2 + Y_1^2 + (\varepsilon + \alpha(\lambda))Y_1Y_2 + Q(Y_1, Y_2, \mu), \end{cases} \tag{4.7}$$

where $\mu_1(\lambda_1, \lambda_2) = \frac{\Upsilon_{00}}{\Upsilon_{20}} - \frac{\Upsilon_{10}^2}{4\Upsilon_{20}^2}$, $\mu_2(\lambda_1, \lambda_2) = \frac{\Upsilon_{01}}{\sqrt{\Upsilon_{20}}} - \frac{\Upsilon_{11}\Upsilon_{00}}{2\Upsilon_{20}^{3/2}}$, $\varepsilon + \alpha(\lambda) = \frac{\Upsilon_{11}}{\sqrt{\Upsilon_{20}}}$, $\alpha(0) = 0$ and $Q(X_1, X_2, 0)$ is a power series in (Y_1, Y_2) with powers $Y_1^i Y_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$. The system (4.7) can be rewritten as:

$$\begin{cases} \frac{dY_1}{d\tau} = Y_2 \\ \frac{dY_2}{d\tau} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2)Y_2 + Y_1^2 + \varepsilon Y_1Y_2 + S(Y_1, Y_2, \mu), \end{cases} \tag{4.8}$$

where $S(Y_1, Y_2, \mu)$ is a power series in Y_1, Y_2, μ_1, μ_2 with powers $Y_1^i Y_2^j \mu_1^k \mu_2^l$ satisfying $i + j + k + l \geq 4$ and $i + j \geq 3$.

As Rank$\left[\frac{\partial(\mu_1, \mu_2)}{\partial(\lambda_1, \lambda_2)}\right] = 2$, system (4.8) is locally topologically equivalent to the normal form of the Bogdanov-Takens bifurcation as given below:

$$\begin{cases} \frac{dZ_1}{dt} = Z_2, \\ \frac{dZ_2}{dt} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2)Z_2 + Z_1^2 \pm Z_1Z_2, \end{cases} \tag{4.9}$$

5. Numerical Simulations

1) If $a = 0.7, b = 0.8, \delta = 0.01, m = 2, h = 0.2$. Then the system (2.3) has two positive interior equilibrium point $E_1^*(x_1^* y_1^*) =$ (0.787085, 0.847733) and $E_2^*(x_2^* y_2^*) = (0.188224, 2.53764)$ and one prey pre equilibrium $E_1 = (0,20)$. If $h = h^{|SN|} = 0.303748$, The two interior equilibrium points coincide and the system (2.3) has only one equilibrium point $\bar{E}(\bar{x}, \bar{y}) = (0.487654, 1.82077)$. If $h = 0.4$ the system (2.3) has no interior equilibrium point (see Figure 1a). The saddle node bifurcation diagram has been depicted in the Figures 1b, 1c and the phase portrait diagram for $h = h^{|SN|} = 0.303748$ is depicted in Figure 1d in which the equilibrium point $\bar{E}$ is stable for the region lie between Magenta color trajectories while unstable for the remaining region.

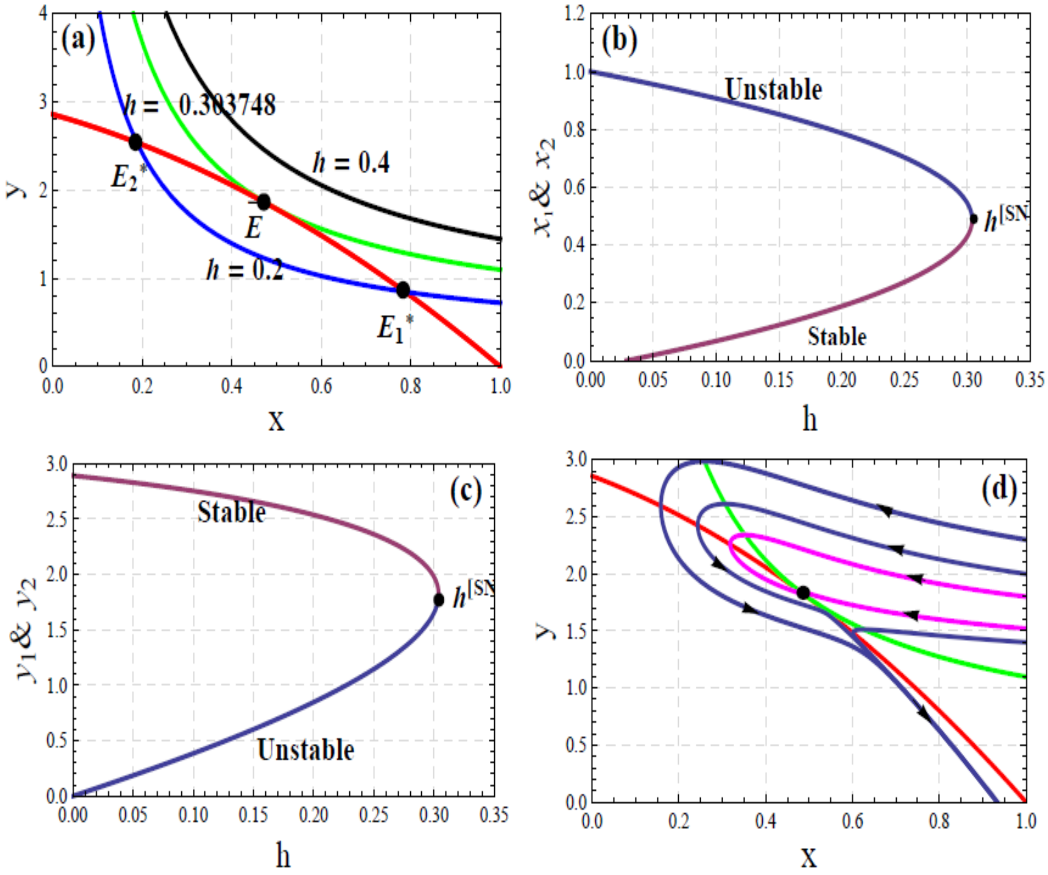

Fig. 1 Diagram (a) shows how the numbers of interior equilibrium point's change with h while keeping other parameters fixed. The red curve is prey nullcline and Blue curve (h = 2), green curve (h = 0.303748) and black curve (h = 0.4) are predator nullcline. (b) and (c) are bifurcation diagrams for the threshold value of the harvesting parameter ($h^{SN} = 0.303748$). (d) Phase portrait diagram of the system (2.3) when h = 0.303748. The other fixed parameters are a = 0.7; b = 0.8; d = 0.01; m = 2.

2) If $a = 0.7, b = 0.8, \delta = 0.01, m = 2, h = 0.2$. Then the point $E_1^*(x_1^* y_1^*)$ is a saddle point and the point $E_2^*(x_2^* y_2^*)$ is stable (see Figure 2a). If $\delta = \delta^{[hf]} = 0.02737585$, the interior equilibrium point are $E_1^*(x_1^* y_1^*) = (0.805989, 0.777705)$ and $E_2^*(x_2^* y_2^*) = (0.127836, 2.65117)$ and the system undergoes to a hopf bifurcation at the point $E_2^*(x_2^* y_2^*)$ and since the first Lyapunove number $\sigma = -22.1349\pi$, a stable limit cycle arises around the point $E_2^*(x_2^* y_2^*)$ (see Figure 2b). If $\delta = 0.03976$, the interior equilibrium point are $E_1^*(x_1^* y_1^*) = (0.817172, 0.735796)$, $E_2^*(x_2^* y_2^*) = (0.0881339, 2.72014)$ and a homoclinic loop is created around E_2^* (see Fig.2c).

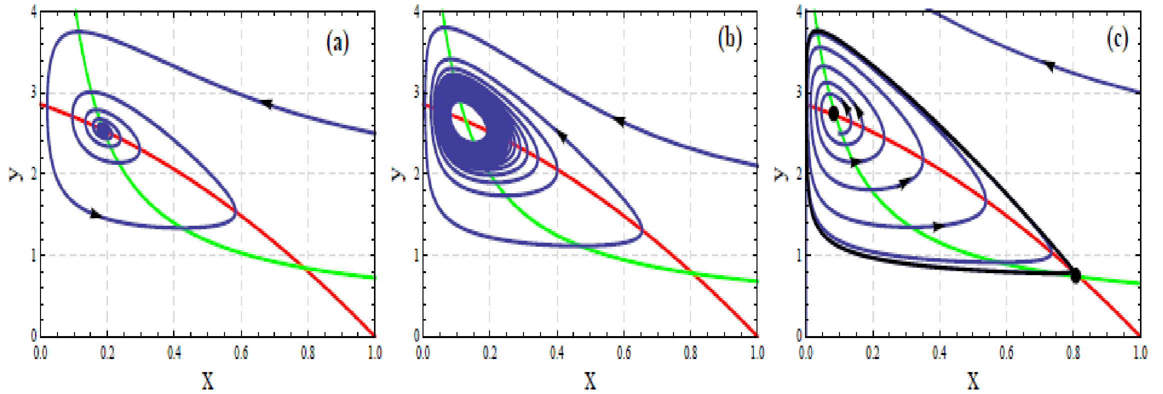

Fig. 2 Diagram (a) is phase portrait diagram of the system (2.3) for fixed parameters a = 0.7; b = 0.8; m = 2.h = 0:2; d = 0.01. The interior equilibrium point E_2^* is an asymptotically stable. (b) a stable limit cycle bifurcates through Hopf bifurcation around the equilibrium point E_2^* for d = 0.02737585. (c) Limit cycle collides with the saddle point E_1^* and a homoclinic loop arises around E_2^*.

3) If $a = 0.7, b = 0.1, m = 3, h = 0.4, \delta = 0.0933333$. Then $ah = \delta m$ and $b + \delta - \delta m = -0.0866667$ the system (2.3) has no interior equilibrium point. The phase portrait diagram has been depicted in Figure 3a. If $b = 0.8$, $ah = \delta m, b + \delta - \delta m = 0.613333$ and the system (2.3) has one positive interior equilibrium point. The phase portrait diagram has been depicted in Figure 3b.

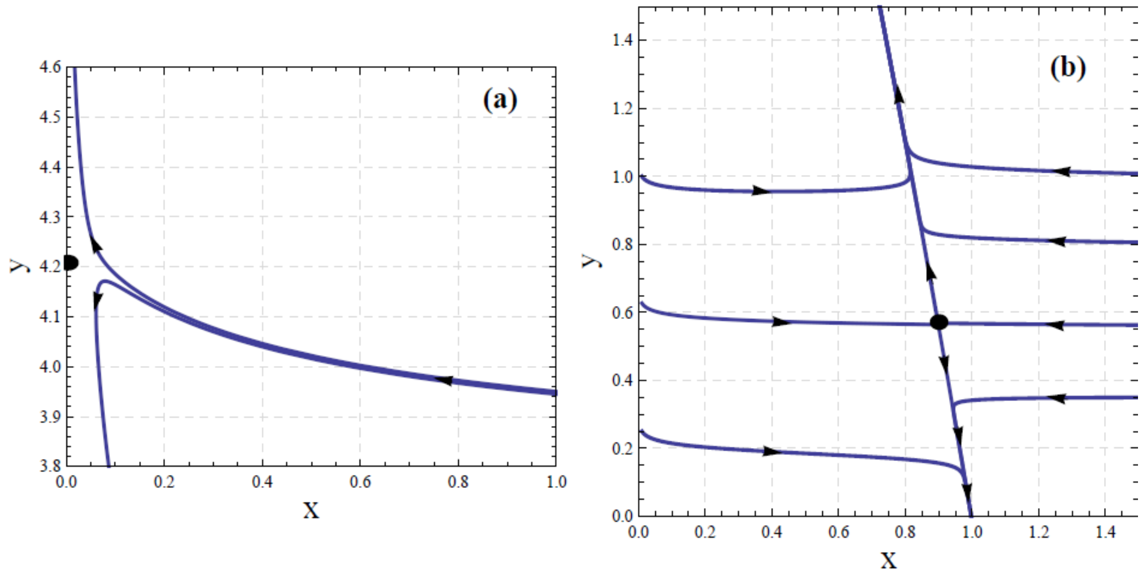

Fig. 3 (a) Phase portrait diagram for the parameters a = 0.7; b = 0.1; m = 3; h = 0.4; d = 0.093333. No interior equilibrium point exists. (b) Phase portrait diagram for the parameters b = 0.8. A unique interior equilibrium point is a saddle.

6. Results and Discussion

A real world problem is modelled using non-linear mathematical equations to describe the process with the help of a suitable number of variables, parameters, and so on. In this paper, we have analysed a non-linear bidimensional predator-prey model with Holling type II functional response in the presence of additional food to predator and constant yield predator harvesting. The qualitative analysis of proposed system shows that the harvesting rate and additional food coefficient affects much on the system. It is shown that the system (2.3) have only one axial equilibrium point while the positive interior equilibrium point changes from zero to two, depends upon the parametric conditions. These conditions are shown in Lemma 3.1 and depicted graphically in Figure 1a. In Theorem 3.1 a it is shown that axial equilibrium point is either an unstable node or a saddle point. If two interior equilibrium points exist, one is always a saddle point while other is either asymptotically stable or a saddle or a weak focus (or a centre) depends upon certain parametric conditions, obtained in theorem 3.1c and depicted in Figures 2a and 2b.

It is also shown that the system (2.3) undergoes hopf bifurcation with respect to hopf bifurcation parameter δ, function of additional food coefficient A. The first Liapanov number is calculated to study the stability of the limit cycle. In example 2 we calculated the Liapunov number numerically which is negative and hence a stable limit cycle arises around the equilibrium point, depicted in Figure 2b. An homoclinic loop is also obtained numerically in Figure 2c. The parametric conditions are obtained under which the system (2.3) enters a Transcritical bifurcation. The phase portrait diagrams of the system (2.3) for these parametric conditions are sketched in Figures 3a and 3b. A parametric region where one of the coexisting equilibrium is saddle and other is stable which gives the existence of a saddle-node bifurcation is obtained. The saddle-node bifurcation diagram is shown in Figures 1b, 1c and the parametric region lying between the Magenta colour trajectories of Figure 1d. The ecological significance of saddle-node and transcritical bifurcation gives the maximum threshold for continuous harvesting and providing additional food without the extinction risk of the predator species.

References

1. *Lotka A., 'Elements of Physical Biology'*, Williams and Williams, Baltimore, *1925.*
2. *Volterra V.*, 'Fluctuations in the abundance of species considered mathematically', *Nature,* ***CXVIII*** *(1926), 558-560.*
3. *Clark C.W., 'Mathematical Bionomics: The Optimal Management of Renewable Resources',* Wiley, New York, *1976.*
4. *Brauer F., Soudack A.C.*, 'Stability regions and transition phenomenon for harvested predator-prey system', *J.Math.Biol.,* ***7*** *(1979), 319-337.*
5. *Brauer F., Soudack A.C.*, 'Stability regions in predator-prey systems with constant-rate prey harvesting', *J.Math. Biol.,* ***8*** *(1979), 55-71.*
6. *Beddington J.R., Cooke J.G.* 'Harvesting from a prey-predator complex', *Ecol. Modelling,* ***14*** *(1982), 155-177.*
7. *Xiao D., Ruan S.*, 'Bogdanov-Takens bifurcations in predator-prey systems with constant rate harvesting', *Fields Inst. Commun.,* ***21*** *(1999), 493*-506.
8. *Xiao D., Jennings L.S.*, 'Bifurcation of a ratio-dependent predator-prey system with constant rate harvesting', *SIAM J. Appl. Math.,* ***65*** *(2005), 737-753.*
9. *Xiao M., Wenxia L., Maoan H.*, 'Dynamics in a ratio-dependent predator-prey model with predator harvesting', *J. Math. Anal. Appl.,* ***324*** *(2006) 14-29.*
10. *Zhu C.R., Lan K .Q.*, 'Phase portraits, Hopf bifurcation and limit cycles of Leslie-Gower predator-prey system with harvesting rates', *Dis. Con. Dyn. Sys. Series B,* ***14*** *(2010), 389-306.*
11. *Gong Y., Huang J.*, 'Bogdanov-Takens bifurcation in a Leslie-Gower predator-prey model with prey harvesting', *Acta Math. Appl. Sin. English series,* ***30*** *(2014), 239-244.*
12. *Gong Y., Huang J., Ruan S., Lou Y.*, 'Bifurcation analysis in predator-prey model with constant yield predator harvesting', *Dis. Con. Dyn. Sys. Series B,* ***18*** *(2013), 2101-2121.*

13. *Holt R.D., Lawton J.H.,* 'The ecological consequences of shared natural enemies', *Ann. Rev. Ecol. Syst.,* ***25*** *(1994), 495-520.*
14. *Baalen M. Van, Krivan V., Rijn P.C.J. Van, Sabelis M.W.,* 'Alternative food, switching predators, and the persistence of predator-prey systems', *Am. Nat.,* ***157*** *(2001), 512-524.*
15. *Wootton G.T.,* 'The nature and consequences of indirect effects in ecological communities', *Ann. Rev. Ecol. Syst.,* ***25*** *(1994), 443-466.*
16. *Harwood J.D., Obrycki J.J.,* 'The role of alternative prey in sustaining predator populations', *In: Hoddle, M.S. (Ed.). Proceedings of Second International Symposium on Biological Control of Arthropods,* ***2*** *(2005), .453-462.*
17. *Spencer P.D., Collie J.S.,* 'A simple predator prey model of exploited marine fish populations in corporating alternative prey', *ICES Journal of Marine Science,* ***53*** *(1996), 615-618.*
18. *Srinivasu P.D.N., Prasad B.S.R.V., Venkatesulu M.,* 'Biological control through provision of additional food to predators: a theoretical study', *Theoretical Population Biology,* ***72*** *(2007), 111-120.*
19. *Sen M., Srinivasu P.D.N., Banerjee M.,* 'Global dynamics of an additional provided predator-prey system with constant harvest in predators', *Appl. Math. Comp.,* ***250*** *(2015), 193-211.*
20. *Kar T.K., Ghosh B.,* 'Sustainability and optimal control of an exploited prey predator system through provision of alternative food to predator', *Bio Systems,* ***109*** *(2012), 220-232.*
21. *Perko L., 'Diffrential Equation and Dynamical Systems',* Springer, New York, *2001.*