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Numerical Solution of two-dimensional Fredholm Integral Equation by Chebyshev and Legendre polynomials using Collocation Method

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Abstract
In this paper we investigate Chebyshev and Legendre polynomials using collocation method that transforming an integral equation to a system of algebraic equations with unknown coefficients. We present two different numerical examples of two dimensional Fredholm linear integral equation. Numerical results and the estimated error are calculated in each example using Maple17.

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Chebyshev and Shifted Chebyshev Polynomials, Legendre and Shifted Legendre Polynomials, Two-dimensional Fredholm Integral Equations, Collocation Method.

1. Introduction

The two-dimensional integral equations provide an important tool for modeling numerous problems in many engineering applications. Recently, many researches have been focused on the development of more advanced and efficient methods for the integral equations [1]-[4], [6], [8]-[10], [13]. Aim of this thesis is to solve two-dimensional Fredholm linear integral equations of the second kind using Chebyshev (first, second, third and fourth kind), Legendre and their shifted polynomials.

In this section we present the main concepts of Chebyshev and Legendre polynomials, with their collocation points. The description of collocation method is explained in section 2 by Chebyshev, Legendre and their shifted polynomials. Two numerical examples are solved in section 3 where we show absolute error of each polynomial with plots. Finally, we discuss the results obtained in section 4.

Consider this two-dimensional Fredholm integral equations of the form:

$$u(x,t) = f(x,t) + \int_{-1}^{1}\int_{-1}^{1} k(x,t,y,z)\,u(y,z)\,dydz \qquad (1.1)$$
$$(x,t) \in [-1,1] \times [-1,1]$$

where $u(x,t)$ is an unknown scalar-valued function, $f(x,t)$ and $k(x,t,y,z)$ are continuous functions on $[-1,1]^2$ and $[-1,1]^4$ respectively.

1.1 Chebyshev Polynomials [7], [14]

In this section we introduce some properties of Chebyshev polynomials which will help us to construct our main results. There are several kinds of Chebyshev polynomials. In particular the first and second kind polynomials $T_n(x)$ and $U_n(x)$ should be introduced, also a pair of related (Jacobi) polynomials $V_n(x)$ and $W_n(x)$ which are called the Chebyshev polynomials of the third and fourth kinds, furthermore the shifted Chebyshev polynomials $T_n^*(x)$, $U_n^*(x)$, $V_n^*(x)$ and $W_n^*(x)$ are covered.

1.1.1 Chebyshev polynomial of first kind

The Chebyshev polynomial $T_n(x)$ of 1^{st} kind is a polynomial in x of degree n , has a recurrence relation:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

where: $T_0(x) = 1$ and $T_1(x) = x$ $\quad n = 2,3\ldots$

The collocation points of $T_n(x)$ on $[-1,1]$ is

$$x_k = \cos\left(\frac{k\pi}{n}\right) \tag{1.2}$$

for: $k = 0,1,2\ldots n$

1.1.2 Shifted Chebyshev polynomial of first kind

The shifted Chebyshev polynomial of 1^{st} kind is defined as:

$$T_n^*(x) = T_n(2x-1)$$

Here the "shifting" function $[x \to 2x-1]$ is chosen such that it bijectively maps the interval $[0,1]$ to the interval $[-1,1]$, indicating that the polynomial $T_n^*(x)$ are orthogonal on $[0,1]$.

The collocation points of $T_n^*(x)$ on $[0,1]$ is:

$$x_k = \frac{1}{2}\left(1 + \cos\left(\frac{k\pi}{n}\right)\right) \tag{1.3}$$

for: $k = 0,1,2\ldots n$

1.1.3 Chebyshev polynomial of second kind

The Chebyshev polynomial $U_n(x)$ of 2^{nd} kind is a polynomial in x of degree n, has a recurrence relation:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

where: $U_0(x) = 1$ and $U_1(x) = 2x$ $\quad n = 2,3\ldots$

The collocation points a of $U_n(x)$ on $[-1,1]$ is:

$$x_k = \cos\frac{(2k+1)\pi}{2(n+1)} \tag{1.4}$$

for: $k = 0,1,2\ldots n$

1.1.4 Shifted Chebyshev polynomial of second kind

The shifted Chebyshev polynomial of 2^{nd} kind is defined as:

$$U_n^*(x) = U_n(2x-1)$$

The collocation points of $U_n^*(x)$ on $[0,1]$ is:

$$x_k = \frac{1}{2}\left(1 + \cos\frac{(2k+1)\pi}{2(n+1)}\right) \tag{1.5}$$

for: $k = 0,1,2\ldots n$

1.1.5 Chebyshev polynomial of third kind

The Chebyshev polynomial $V_n(x)$ of 3^{rd} kind is a polynomial in x of degree n, has a recurrence relation:

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x)$$

where: $V_0(x) = 1$ and $V_1(x) = 2x - 1$ $\quad n = 2,3\ldots$

The collocation points of $V_n(x)$ on $[-1,1]$ is:

$$x_k = \cos\frac{2k\pi}{2n+1} \tag{1.6}$$

for: $k = 0,1,2...n$

1.1.6 Shifted Chebyshev polynomial of third kind

The shifted Chebyshev polynomial of 3^{rd} kind is defined as:

$$V_n^*(x) = V_n(2x-1)$$

The collocation points of $V_n^*(x)$ on $[0,1]$ is:

$$x_k = \frac{1}{2}\left(1+\cos\frac{2k\pi}{2n+1}\right) \tag{1.7}$$

for: $k = 0,1,2...n$

1.1.7 Chebyshev polynomial of forth kind

The Chebyshev polynomial $W_n(x)$ of 4^{th} kind is a polynomial in x of degree n, has a recurrence relation:

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x)$$

where: $W_o(x) = 1$ and $W_1(x) = 2x+1$ $\qquad n = 2,3...$

The collocation points of $W_n(x)$ on $[-1,1]$ is:

$$x_k = \cos\frac{(2k+1)\pi}{2n+1} \tag{1.8}$$

for: $k = 0,1,2...n$

1.1.8 Shifted Chebyshev polynomial of fourth kind

The shifted Chebyshev polynomial of 4^{th} kind is defined as:

$$W_n^*(x) = W_n(2\mathbf{x}-1)$$

The collocation points of $W_n^*(x)$ on $[0,1]$ is:

$$x_k = \frac{1}{2}\left(1+\cos\frac{(2k+1)\pi}{2n+1}\right) \tag{1.9}$$

for: $k = 0,1,2...n$

All the above for $x = \cos\theta$. If the range of the variable x is the interval $[-1,1]$, then the range of the corresponding variable θ can be taken as $[0,1]$. These ranges are traversed in opposite directions, since $x = -1$ corresponds to $\theta = \pi$ and $x = 1$ corresponds to $\theta = 0$.

1.2 Legendre Polynomials [5, 11, 12]

The Legendre polynomials were first introduced in 1782 by Adrien-Marie Legendre as the coefficients in the expansion of the Newtonian potential:

$$\frac{1}{|x-x'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} = \sum_{m=0}^{\infty}\frac{r'^m}{r^{m+1}}P_m(\cos\gamma)$$

where r and r' are the lengths of the vectors x and x' respectively and γ is the angle between those two vectors.

1.2.1 Legendre polynomial of first kind

The Legendre polynomials, sometimes called Legendre functions of the first kind. This is given by the following expression, known as Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

for: $-1 \le x \le 1$ and $n \ge 0$

1.2.2 Shifted Legendre polynomial of first kind

The shifted Legendre polynomials are defined as:

$$\tilde{P}_n(x) = P_n(2x-1)$$

Here the "shifting" function $[x \to 2x-1]$ is chosen such that it bijectively maps the interval $[0,1]$ to the interval $[-1,1]$, implying that the polynomials $\tilde{P}_n(x)$ are orthogonal on $[0,1]$.

The analogue of Rodrigues' formula for the shifted Legendre polynomials is:

$$\tilde{P}_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^2-x)^n$$

for: $0 \le x \le 1$ and $n \ge 0$

The collocation points of $P_n(x)$, $\tilde{P}_n(x)$ on $[a,b]$ is given by:

$$x_k = a + \frac{(b-a)k}{n} \tag{1.10}$$

for: $k = 0,1,2...n$ and $a \le x \le b$

2. Solution of two dimensional integral equations using Chebyshev and Legendre polynomials by collocation method

This method is based on approximating the unknown function $u(x,t)$ as:

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} G_i(x) G_j(t) \tag{2.1}$$

where $(x,t) \in [-1,1] \times [-1,1]$ and G is one of Chebyshev polynomials or Legendre $[(x,t) \in [0,1] \times [0,1]$ if G is a shifted function] and a_{ij} are constants to be determined.

If the infinite series (2.1) is truncated then it can be written as:

$$u(x,t) = u_n(x,t) \approx \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} G_i(x) G_j(t) \tag{2.2}$$

substituting from equation (2.2) into equation (1.1) we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left[G_i(x)G_j(t) - \int_{-1}^{1}\int_{-1}^{1} k(x,t,y,z) G_i(y) G_j(z)\, dydz\right] = f(x,t). \tag{2.3}$$

hence the residual equation is defined as:

$$R_N(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left[G_i(x)G_j(t) - \int_{-1}^{1}\int_{-1}^{1} k(x,t,y,z) G_i(y) G_j(z)\, dydz\right] - f(x,t) = 0. \tag{2.4}$$

using the collocation points x_r, t_s according to each kind of equations (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), (1.8), (1.9) and (1.10). Equation (2.4) can be written as:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left[G_i(x_r)G_j(t_s) - \int_{-1}^{1}\int_{-1}^{1} k(x_r,t_s,y,z) G_i(y) G_j(z)\, dydz\right] = f(x_r,t_s). \tag{2.5}$$

Hence we get a system of $(N+1)^2$ linear algebraic equations contain $(N+1)^2$ of a_{ij} unknown coefficients. Solving this system using maple, we obtain the value of these a_{ij} such that

$i, j = 0, 1, 2 \ldots N$. Substituting from these values into equation (2.2) we obtain the approximate solution.

3. Numerical Examples:

In this section some numerical examples of two dimensional linear Fredholm integral equation are presented to illustrate the methods. All results are obtained using Maple17.

Example 1:

Consider the following two-dimensional Fredholm integral equation [16]:

$$u(x,t) = x^2 \sin(t) - \frac{4}{3} t(\sin(1) - \cos(1)) + \int_{-1}^{1}\int_{-1}^{1} (xy + tz)\, u(y,z)\, dydz \tag{3.1}$$

$x, t \in [-1, 1]$ with the exact solution $u(x,t) = x^2 \sin(t)$

Solution

Taking N=6, then substituting into equation (2.2), we obtain

$$u(x,t) = \sum_{i=1}^{6}\sum_{j=1}^{6} a_{ij} G_i(x) G_j(t) \tag{3.2}$$

substituting from equation (3.2) into (3.1) and using collocation points x_r, t_s we get:

$$\sum_{i=1}^{6}\sum_{j=1}^{6} a_{ij} \left[G_i(x_r) G_j(t_s) - \int_{-1}^{1}\int_{-1}^{1} (x_r y + t_s z) G_i(y) G_j(z)\, dydz \right] = x_r^2 \sin(t_s) - \frac{4}{3} t_s (\sin(1) - \cos(1)) \tag{3.3}$$

we get a system of 49 linear algebraic equations contains 49 of a_{ij} unknown coefficients.

1) Applying Chebyshev polynomial of 1^{st} kind, using collocation points (1.2) we obtain:

$$x_0 = t_0 = 1, \quad x_1 = t_1 = \frac{\sqrt{3}}{2}, \quad x_2 = t_2 = \frac{1}{2}, \quad x_3 = t_3 = 0,$$
$$x_4 = t_4 = -\frac{1}{2}, \quad x_5 = t_5 = -\frac{\sqrt{3}}{2}, \quad x_6 = t_6 = -1.$$

2) Applying Chebyshev polynomial of 2^{nd} kind, using collocation points (1.4) we obtain:

$$x_0 = t_0 = 0.9749279122, \quad x_1 = t_1 = 0.7818314824, \quad x_2 = t_2 = 0.4338837394, \quad x_3 = t_3 = 0,$$
$$x_4 = t_4 = -0.4338837394, \quad x_5 = t_5 = -0.7818314824, \quad x_6 = t_6 = -0.9749279122.$$

3) Applying Chebyshev polynomial of 3^{rd} kind, using collocation points (1.6) we obtain:

$$x_0 = t_0 = 1, \quad x_1 = t_1 = 0.8854560257, \quad x_2 = t_2 = 0.5680647466, \quad x_3 = t_3 = 0.1205366804,$$
$$x_4 = t_4 = -0.3546048868, \quad x_5 = t_5 = -0.7485107480, \quad x_6 = t_6 = -0.9709418174.$$

4) Applying Chebyshev polynomial of kind, using collocation points (1.6) we obtain:

$$x_0 = t_0 = 0.9709418174, x_1 = t_1 = 0.7485107480, x_2 = t_2 = 0.3546048868, x_3 = t_3 = -0.1205366804,$$
$$x_4 = t_4 = -0.568064766, x_5 = t_5 = -0.8854560257, \quad x_6 = t_6 = -1.$$

5) Applying Legendre polynomial, using collocation points (1.10) we obtain:

$$x_0 = t_0 = -1, \quad x_1 = t_1 = -\frac{2}{3}, \quad x_2 = t_2 = -\frac{1}{3}, \quad x_3 = t_3 = 0,$$
$$x_4 = t_4 = \frac{1}{3}, \quad x_5 = t_5 = \frac{2}{3}, \quad x_6 = t_6 = 1.$$

By solving each of these systems similarly using collocation points that have been calculated, we get the approximate solution of eq. (3.1). The numerical results are shown in Tables 1, 2 and Figures 1, 2, 3, 4, 5, and 6.

Example 2:
Consider the following two-dimensional Fredholm integral equation [15]:

$$u(x,t) = xe^{-t} - \frac{1}{2}t - \frac{7}{12}x + \frac{1}{3}xe^{-1} + \int_0^1\int_0^1 (xy + te^z)u(y,z)\,dydz \tag{3.4}$$

$x,t \in [0,1]$ with the exact solution $u(x,t) = xe^{-t} + t$

Solution
Taking N=10, Substituting into equation (2.2), we obtain:

$$u(x,t) = \sum_{i=1}^{10}\sum_{j=1}^{10} a_{ij} G_i(x) G_j(t) \tag{3.5}$$

substituting from equation (3.5) into (3.4) and using collocation points x_r, t_s we get:

$$u(x,t) = \sum_{i=1}^{10}\sum_{j=1}^{10} a_{ij}\left[G_i(x_r)G_j(t_s) - \int_0^1\int_0^1 (x_r y + t_s e^z) G_i(y) G_j(z)\,dydz\right] = x_r e^{-t_s} - \frac{1}{2}t_s - \frac{7}{12}x_r + \frac{1}{3}x_r e^{-1} \tag{3.6}$$

So, we get a system of 121 linear algebraic equations contains 121 of a_{ij} unknown coefficients.

By solving this system similarly to previous example, using collocation points of equations (1.3), (1.5), (1.7), (1.9), (1.10) respectively, we get the approximate solution of eq. (3.4) and the numerical results are shown in Tables 3,4 and Figures 7, 8, 9, 10, 11 and 12.

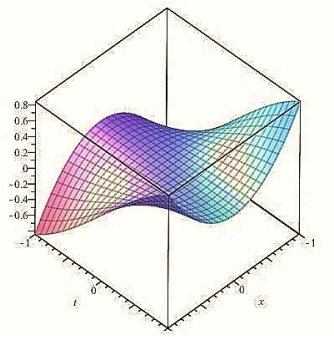

Fig.1: Exact solution of ex.1

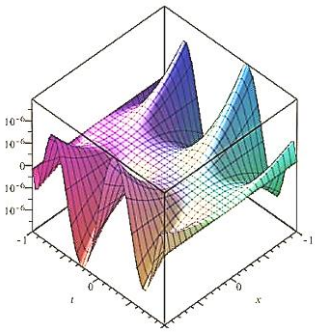

Fig.2: Absolute error of ex.1 for Chebyshev 1st kind

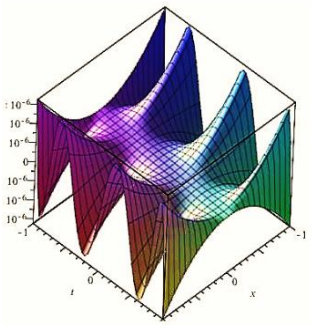

Fig.3: Absolute error of ex.1 for Chebyshev 2nd kind

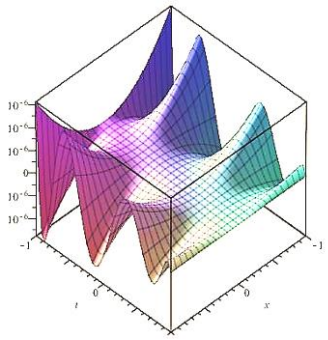

Fig.4: Absolute error of ex.1 for Chebyshev 3rd kind

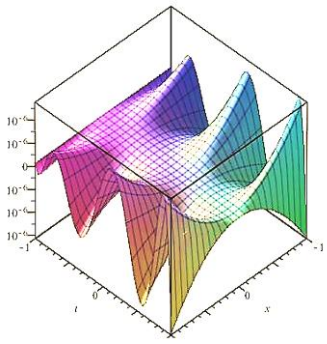

Fig.5: Absolute error of ex.1 for Chebyshev 4th kind

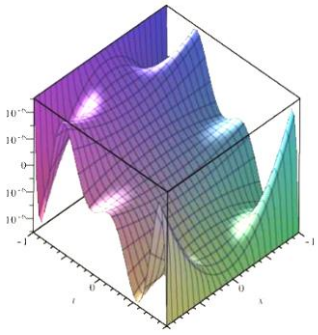

Fig.6: Absolute error of ex.1 for Legendre 1st kind

Table 1: Exact [16] and approximate solutions of example (1) by Chebyshev (first, second, third, fourth kinds) and Legendre polynomials using collocation method N = 6

(x, t)	Exact solution	Approximate solution				
		Chebyshev				Legendre
		1^{st}kind	2^{nd}kind	3^{rd}kind	4^{th}kind	1^{st} kind
(-1, -1)	-0.8414709	-0.8414707	-0.84147423	-0.00099832	-0.84147125	-0.84147604
(-0.9, -0.9)	-0.6344947	-0.6344936	-0.63449261	-0.00794666	-0.63449481	-0.63449255
(-0.8, -0.8)	-0.4591078	-0.4591092	-0.45910771	-0.02659674	-0.45910915	-0.45910827
(-0.7, -0.7)	-0.3156666	-0.3156686	-0.31566796	-0.06230729	-0.31566787	-0.31566962
(-0.6, -0.6)	-0.2032713	-0.2032723	-0.20327250	-0.11985765	-0.20327169	-0.20327488
(-0.5, -0.5)	-0.1198564	-0.1198562	-0.11985689	-0.20327333	-0.11985614	-0.11985940
(-0.4, -0.4)	-0.0623069	-0.0623063	-0.06230692	-0.31566805	-0.06230654	-0.06230910
(-0.3, -0.3)	-0.0265968	-0.0265962	-0.02659667	-0.45910627	-0.02659660	-0.02659830
(-0.2, -0.2)	-0.0079468	-0.0079465	-0.00794671	-0.63449040	-0.00794675	-0.00794774
(-0.1, -0.1)	-0.0009983	-0.0009983	-0.00099834	-0.84147724	-0.00099836	-0.00099883
(0,0)	0	0	8×10^{-24}	8.992×10^{-11}	-1×10^{-10}	0
(0.1,0.1)	0.0009983	0.0009983	0.00099834	0.00099837	0.000998316	0.00099883
(0.2,0.2)	0.0079468	0.0079465	0.00794671	0.00794675	0.007946664	0.00794774
(0.3,0.3)	0.0265968	0.0265962	0.02659667	0.02659660	0.026596737	0.02659830
(0.4,0.4)	0.0623069	0.0623063	0.06230692	0.06230654	0.062307294	0.06230910
(0.5,0.5)	0.1198564	0.1198562	0.11985689	0.11985614	0.119857648	0.11985940
(0.6,0.6)	0.2032713	0.2032723	0.20327250	0.20327169	0.203273326	0.20327488
(0.7,0.7)	0.3156666	0.3156686	0.31566796	0.31566787	0.315668054	0.31566962
(0.8,0.8)	0.4591078	0.4591092	0.45910771	0.45910913	0.459106267	0.45910827
(0.9,0.9)	0.6344947	0.6344936	0.63449261	0.63449481	0.634490402	0.63449255
(1,1)	0.8414709	0.8414707	0.84147423	0.84147125	0.841477238	0.84147604

Table 2: Absolute error results of example (1) by Chebyshev (first, second, third, fourth kinds) with N=6.

(x, t)	Absolute Error				
	Chebyshev				Legendre 1^{st} kind
	1^{st}kind	2^{nd}kind	3^{rd}kind	4^{th}kind	
(-1, -1)	2.979×10^{-7}	0.0000032484	0.0000062536	2.656×10^{-7}	0.0000050555
(-0.9, -0.9)	0.000001156070	0.00000218490	0.0000043947	1.51681×10^{-8}	0.00000224399
(-0.8, -0.8)	0.000001273970	1.8465×10^{-7}	0.0000016316	0.000001250558	3.7560×10^{-7}
(-0.7, -0.7)	0.0000018835116	0.00000129646	0.0000013872	0.000001200468	0.00000295655
(-0.6, -0.6)	9.599271×10^{-7}	0.00000121895	0.0000020360	4.02888×10^{-7}	0.00000359456
(-0.5, -0.5)	1.490525×10^{-7}	5.09019×10^{-7}	0.0000012635	2.41291×10^{-7}	0.000003017706
(-0.4, -0.4)	6.6881881×10^{-7}	1.8983×10^{-8}	3.5915×10^{-7}	3.92973×10^{-7}	0.000002167047
(-0.3, -0.3)	5.89507465×10^{-7}	1.50879×10^{-7}	8.108×10^{-8}	2.17864×10^{-7}	0.000001480996
(-0.2, -0.2)	2.80485475×10^{-8}	6.65538×10^{-8}	1.0886×10^{-7}	2.2864×10^{-7}	9.682475×10^{-7}
(-0.1, -0.1)	6.37420336×10^{-8}	6.88268×10^{-9}	1.7854×10^{-8}	3.2227×10^{-7}	4.971057×10^{-7}
(0,0)	0	-8×10^{-24}	1.5692×10^{-10}	1.09×10^{-7}	0
(0.1,0.1)	6.37420336×10^{-8}	6.88268×10^{-9}	3.2149×10^{-8}	1.7888×10^{-7}	4.971057×10^{-7}
(0.2,0.2)	2.80485475×10^{-8}	6.65538×10^{-8}	2.282×10^{-8}	1.088681×10^{-7}	9.682475×10^{-7}
(0.3,0.3)	5.89507465×10^{-7}	1.50879×10^{-7}	2.1784×10^{-7}	8.10584×10^{-7}	0.000001480996
(0.4,0.4)	6.6881881×10^{-7}	1.8983×10^{-8}	3.9292×10^{-7}	3.592319×10^{-7}	0.000002167047
(0.5,0.5)	1.490525×10^{-7}	5.09019×10^{-7}	2.414×10^{-7}	0.000001263291	0.000003017706
(0.6,0.6)	9.599271×10^{-7}	0.00000121895	4.027×10^{-7}	0.00000203601	0.00000359456
(0.7,0.7)	0.0000018835116	0.00000129646	0.0000012005	0.00000138726	0.00000295655
(0.8,0.8)	0.000001273970	1.8465×10^{-7}	0.0000012505	0.00000163156	3.7560×10^{-7}
(0.9,0.9)	0.000001156070	0.00000218490	1.54×10^{-8}	0.00000439389	0.00000224399
(1,1)	2.979×10^{-7}	0.0000032484	2.651×10^{-7}	0.0000062519	0.0000050555

Table 3: Exact [15] and approximate solutions of example (2) by shifted Chebyshev (first, second, third, fourth kinds) and shifted Legendre polynomials using collocation method with N=10.

(x, t)	Exact solution	Approximate solution				
		Shifted Chebyshev				Shifted Legendre 1^{st}kind
		1^{st}kind	2^{nd}kind	3^{rd}kind	4^{th}kind	
(0,0)	0	10^{-10}	$-3.347\ 10^{-8}$	0.00001478	$5.0021\ 10^{-11}$	$-1.273\ 10^{-10}$
(0.1,0.1)	0.19048374	0.1904837441	0.19048367	0.19048597	0.19048359	0.19048374
(0.2,0.2)	0.36374615	0.3637461534	0.36374630	0.36374674	0.36374576	0.36374615
(0.3,0.3)	0.52224547	0.5222454710	0.52224559	0.52224384	0.52224483	0.52224547
(0.4,0.4)	0.66812802	0.6681280240	0.66812792	0.66812372	0.66812718	0.66812802
(0.5,0.5)	0.80326533	0.8032653358	0.80326537	0.80326064	0.80326428	0.80326533
(0.6,0.6)	0.92928698	0.9292869878	0.92928732	0.92928181	0.92928586	0.92928698
(0.7,0.7)	1.04760971	1.047609719	1.04760977	1.04760276	1.04760874	1.04760971
(0.8,0.8)	1.15946317	1.159463181	1.15946264	1.15945266	1.15946231	1.15946317
(0.9,0.9)	1.26591269	1.265912704	1.26591267	1.26590286	1.26591148	1.26591269
(1,1)	1.36787944	1.367879455	1.36788023	1.36786842	1.36787815	1.36787944

Table 4:Absolute error results of example (2) by shifted Chebyshev (first, second, third, fourth kinds) and shifted Legendre polynomial methods N=10

(x, t)	Absolute Error				
	Shifted Chebyshev				Shifted Legendre 1^{st} kind
	1^{st}kind	2^{nd}kind	3^{rd}kind	4^{th}kind	
(0,0)	$1.5952\ 10^{-10}$	3.3421×10^{-8}	1.478×10^{-5}	5.1278×10^{-11}	4.2506×10^{-10}
(0.1,0.1)	2.1824×10^{-9}	7.6310×10^{-8}	2.231×10^{-6}	1.5606×10^{-7}	2.2159×10^{-10}
(0.2,0.2)	2.6528×10^{-9}	1.4571×10^{-7}	5.9023×10^{-7}	3.9540×10^{-7}	2.2159×10^{-11}
(0.3,0.3)	4.7884×10^{-9}	1.2218×10^{-7}	1.6289×10^{-6}	6.3798×10^{-7}	1.6541×10^{-10}
(0.4,0.4)	5.6433×10^{-9}	9.8439×10^{-8}	4.3011×10^{-6}	8.3534×10^{-7}	9.5619×10^{-11}
(0.5,0.5)	6×10^{-9}	4.4809×10^{-8}	4.6898×10^{-6}	1.0525×10^{-6}	4.7080×10^{-11}
(0.6,0.6)	6.1488×10^{-9}	3.3371×10^{-7}	5.1660×10^{-6}	1.1212×10^{-6}	1.6591×10^{-10}
(0.7,0.7)	6.9727×10^{-9}	6.3136×10^{-8}	6.9584×10^{-6}	9.7644×10^{-7}	2.9089×10^{-11}
(0.8,0.8)	9.6753×10^{-9}	5.3222×10^{-7}	1.0507×10^{-5}	8.5979×10^{-7}	1.6573×10^{-10}
(0.9,0.9)	1.1011×10^{-8}	1.9076×10^{-8}	9.8293×10^{-6}	1.2155×10^{-6}	4.9164×10^{-11}
(1,1)	1.45×10^{-8}	7.799×10^{-7}	1.1019×10^{-5}	1.2851×10^{-6}	3×10^{-10}

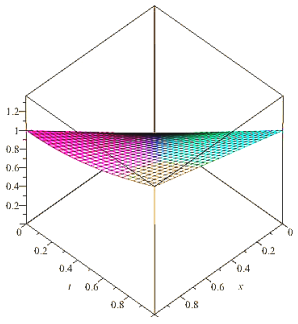

Fig.7: Exact solution of ex.2.

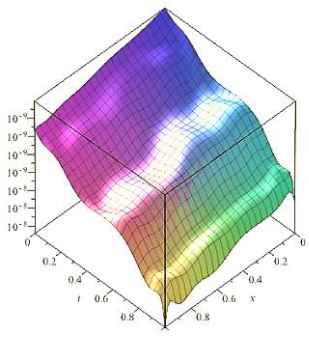

Fig.8: Absolute error of ex.2 for shifted Chebyshev 1^{st} kind.

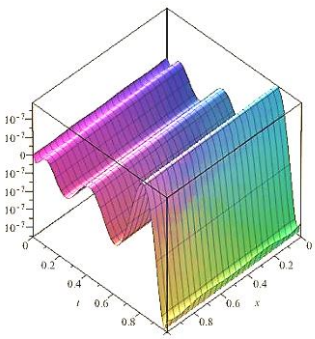

Fig.9: Absolute error of ex.2 for shifted Chebyshev 2^{nd} kind.

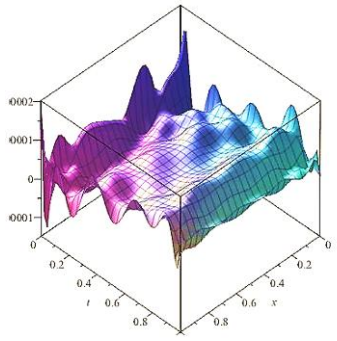

Fig.10:Absolute error of ex.2 for shifted Chebyshev 3rd kind.

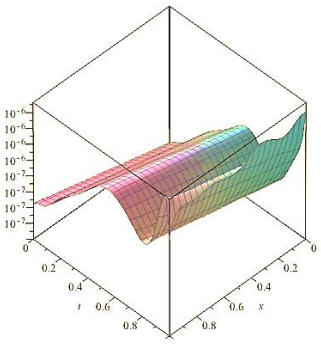

Fig.11: Absolute error of ex.2 for shifted Chebyshev 4th kind.

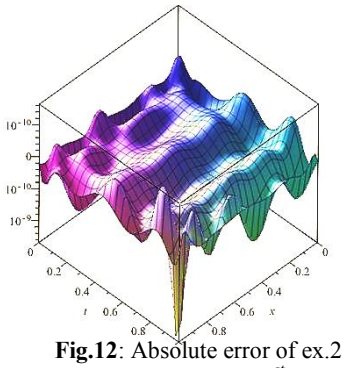

Fig.12: Absolute error of ex.2 for shifted Legendre 1st kind.

4. Conclusions

Two-dimensional Fredholm integral equations are usually difficult to be solved analytically. In many cases, it is required to obtain the approximate solutions, for this purpose the presented method using different polynomials can be proposed. We have investigated the Chebyshev polynomial of four kinds and Legendre polynomial of 1st kind by collocation method for solving the linear two-dimensional Fredholm integral equations.

This technique is innovative. By comparing the results, we find that Chebyshev polynomial of 1st kind is the best among the other mentioned methods above that have range [-1, 1]. Also shifted Legendre polynomial of 1st kind is the best among the other mentioned methods above that have range [0, 1].

The current method can be extended to two-dimensional nonlinear Fredholm integral equations, but there will be some problematic in case of large (N).

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